Analele Universităţii Oradea
Fasc. Matematica, Tom XXXII (2025), Issue No. 2, 5–30

NONLINEAR DISCRETE NEUMANN PROBLEM INVOLVING $p(k)$-LAPLACIAN TYPE OPERATOR

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Abstract. In this paper, we prove the existence and multiplicity of solutions for a discrete nonlinear Neumann problem involving a $p(k)$-Laplacian operator in a T-dimensional Banach space. The technical approach is based on critical point theory and variational methods.

1. Introduction

In this paper, we study the existence and multiplicity of solutions for a discrete nonlinear Neumann problem of the following $p(k)$-Laplacian operator

$$\begin{cases} -\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)) + q(k)|u(k)|^{p(k)-2}u(k) \\ \quad = \lambda f(k,u(k)), \;\; k \in \mathbb{Z}(1,T), \\ \triangle u(0) = \triangle u(T) = 0, \end{cases} \tag{1.1}$$

where $T \geq 2$ is a fixed positive integer, $\mathbb{Z}(a,b)$ denotes the discrete interval $\{a, a+1, \ldots, b-1, b\}$ with a and b integers such that $a < b$, $\triangle u(k) = u(k+1) - u(k)$ is the forward difference operator and $p : \mathbb{Z}(0,T) \to (1,\infty)$, $q : \mathbb{Z}(1,T) \to (1,\infty)$ are given functions, $\lambda > 0$ is a real parameter and $f : \mathbb{Z}(1,T) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to the second variable. Moreover, the function $a(k,\cdot) : [0,\infty) \to [0,\infty)$ is continuous for all $k \in \mathbb{Z}(0,T)$.

Throughout this paper, we denote

$$p^+ := \max_{k\in\mathbb{Z}(0,T)} p(k), \quad p^- := \min_{k\in\mathbb{Z}(0,T)} p(k),$$

$$\overline{q} := \max_{k\in\mathbb{Z}(1,T)} q(k), \quad \underline{q} := \min_{k\in\mathbb{Z}(1,T)} q(k), \quad Q := \sum_{k=1}^{T} q(k).$$

For the function a, we assume the following.

$(H1)$ There exist $a_1 : \mathbb{Z}(0,T) \to [0,\infty)$ and a constant $a_2 > 0$ such that

$$|a(k,|\xi|)\xi| \leq a_1(k) + a_2|\xi|^{p(k)-1},$$

for all $k \in \mathbb{Z}(0,T)$ and $\xi \in \mathbb{R}$.

2020 *Mathematics Subject Classification.* 47A75, 35B38, 35P30, 34L05, 34L30.

Key words and phrases. Discrete boundary value problem, multiple solutions, variational methods, critical point theory, Neumann problem.

$(H2)$ For all $k \in \mathbb{Z}(0,T)$ and $\xi > 0$, one has
$$0 \le a(k,|\xi|)\xi^2 \le p^+ \int_0^{|\xi|} a(k,s)s\,ds.$$
$(H3)$ There exists a positive constant c such that
$$\min\left\{a(k,|\xi|),\ |\xi|\frac{\partial a}{\partial \xi}(k,|\xi|) + a(k,|\xi|)\right\} \ge c|\xi|^{p(k)-2},$$
for all $k \in \mathbb{Z}(0,T)$ and $\xi \in \mathbb{R}$.

Next, we introduce $A_0 : \mathbb{Z}(1,T) \times [0,\infty) \to [0,\infty)$ defined by
$$A_0(k,t) = \int_0^t a(k,\xi)\xi\,d\xi.$$

Remark 1.1. As examples of functions A_0 and a satisfying the above assumptions, we can give the following.

(1) If we let
$$a(k,|\xi|) = |\xi|^{p(k)-2}, \text{ for all } (k,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R},$$
then
$$A_0(k,|\xi|) = \frac{1}{p(k)}|\xi|^{p(k)}.$$

(2) If we put
$$a(k,|\xi|) = \left(1+|\xi|^2\right)^{\frac{p(k)-2}{2}}, \text{ for all } (k,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R},$$
then
$$A_0(k,|\xi|) = \frac{1}{p(k)}\left[\left(1+|\xi|^2\right)^{\frac{p(k)}{2}} - 1\right].$$

(3) If we set
$$a(k,|\xi|) = \left(1 + \frac{|\xi|^{p(k)}}{\sqrt{1+|\xi|^{2p(k)}}}\right)|\xi|^{p(k)-2}, \text{ for all } (k,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R},$$
then
$$A_0(k,|\xi|) = \frac{1}{p(k)}|\xi|^{p(k)} + \frac{\sqrt{1+|\xi|^{2p(k)}}}{p(k)}.$$

Differential equations with variable exponent have been widely used to model many phenomena arising from the study of elastic mechanics [40], electrorheological fluids [36, 37] and image restoration [17]. In recent years, many authors have discussed the existence and multiplicity of solutions for difference equations with boundary value conditions using fixed point theory, lower and upper solutions method, Rabinowitz's global bifurcation theorem, variational methods, critical point theory, etc. We refer the readers to the monograph of Agarwal [1] and the papers [3, 6, 24]. In 2003, variational methods were employed to study difference equations [23], by which various results are investigated. We refer the readers to the recent work [18] for the applications of variational methods on $p(k)$-difference equations.

In [12], Candito and D'Aguí investigated, by using an abstract local minimum theorem due to Bonanno et *al.* [8] and truncation techniques, the existence of constant-sign solutions for a nonlinear Neumann boundary value problem involving the discrete p-Laplacian. In [38], Tian and Ge established the existence of solutions for a second-order discrete Neumann problem with a p-Laplacian. For discrete problems with $p(k)$-Laplacian operator, Gao in [20] using Rabinowitz's global bifurcation theorem, studied the existence of positive solutions for the discrete boundary value problems. Guiro et *al.* in [21] proved, by using the minimization method, the existence and uniqueness of solutions for a family of discrete boundary value problems whose right-hand side belongs to a discrete Hilbert space. Bereanu et *al.* in [5] undertake the existence of periodic or Neumann solutions for the discrete $p(k)$-Laplacian.

In particular, the problem of type (1.1) has been previously studied, for instance, in [31, 32, 34] by using various methods. For example Moussa et *al.* in [31], by using variational methods, have investigated the existence and multiplicity of solutions for the problem (1.1) with $q(k) = 0$ for all $k \in \mathbb{Z}(1, T)$ and Robin boundary conditions $\triangle u(0) = u(T + 1) = 0$. In [32], the present authors studied a more general version of problem (1.1) with heteroclinic condition at the boundary. In [34], by using variational methods and critical point theory, the present authors studied the existence and multiplicity of weak solutions for discrete Kirchhoff-type equations with Dirichlet boundary conditions in T-dimensional Banach space.

The significance of problem (1.1) stems from the presence of an nonhomogeneous differential operator of the form

$$\triangle(a(k-1, |\triangle u(k-1)|)\triangle u(k-1)),$$

where $a(k,\cdot)$ satisfies $(H1)$-$(H3)$. This kind of operator was recently studied in [33]. It generalizes the usual operators with variable exponent.
Indeed, if we take $a(k, |\xi|) = |\xi|^{p(k)-2}$ in the problem (1.1), then we obtain the standard $p(k)$-Laplace difference operator, that is,

$$\triangle_{p(k-1)} u(k-1) := \triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right). \tag{1.2}$$

When $a(k, |\xi|) = \left(1 + |\xi|^2\right)^{\frac{p(k)-2}{2}}$, it corresponds to the generalized mean curvature operator

$$\triangle\left(\left(1 + |\triangle u(k-1)|^2\right)^{\frac{p(k-1)-2}{2}}\triangle u(k-1)\right). \tag{1.3}$$

When

$$a(k, |\xi|) = \left(1 + \frac{|\xi|^{p(k)}}{\sqrt{1 + |\xi|^{2p(k)}}}\right)|\xi|^{p(k)-2},$$

the operator appearing in (1.1) corresponds to a $p(k)$-Laplacian-like operator, given by

$$\triangle\left(\left(1 + \frac{|\triangle u(k-1)|^{p(k-1)}}{\sqrt{1 + |\triangle u(k-1)|^{2p(k-1)}}}\right)|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right). \tag{1.4}$$

Discrete boundary value problems have been intensively studied in the last decade. For the recent papers involving the discrete $p(k)$-Laplacian operator, we refer the readers to [5, 19, 27, 35]. In the case where $p(k)$ is a constant (called the discrete p-Laplacian operator), we refer the readers to the following recent works [2, 13, 14, 15]

and references therein. The discrete $p(k)$-Laplacian operator has more complicated nonlinearities than the discrete p-Laplacian operator, for example, it is not homogeneous. The difference equations involving nonhomogeneous difference operators of type (1.2) were initiated by Mihăilescu et *al.* in [28], where some eigenvalue problems were investigated.
Moreover, Barghouthe et *al.* in [4] established the existence and multiplicity of solutions for (1.4) by using variational methods and critical point theory (see also [30]). In [26], Koné and Ouaro proved, by using the minimization method, the existence and uniqueness of weak solutions for anisotropic discrete boundary value problems.

However, little work has been done referring to a nonlinear Neumann boundary value problem involving the discrete $p(k)$-Laplacian.

Problem (1.1) can be seen as a discrete variant of the variable exponent anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \left|\frac{\partial u}{\partial x_i}\right|\right) \frac{\partial u}{\partial x_i} + q(x)|u|^{p_i(x)-2}u = \lambda f(x,u) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $f \in C\left(\overline{\Omega} \times \mathbb{R}, \mathbb{R}\right)$ a given function satisfying certain properties, $q(x) \geq 1$ continuous on $\overline{\Omega}$ for all $x \in \Omega$, p_i continuous on $\overline{\Omega}$, $\displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-} > 1$ and $1 < p_i(x) < N$ for all $x \in \overline{\Omega}$ and all $i \in \mathbb{Z}(1,T)$, where $p_i^- := \inf_{x \in \Omega} p_i(x)$, $\lambda > 0$ a real number.

Recently, I. H. Kim and Y. H. Kim [25] studied problem (1.5) with $q(k) = 0$ under homogeneous Dirichlet boundary condition ($u = 0$ on $\partial\Omega$).

In this paper, we investigate the existence and multiplicity of solutions for the problem (1.1), using variational methods and critical point theory.

This work is motivated by the paper [12] and is organized as follows. In Section 2, we establish the variational framework associated with problem (1.1). Some necessary preliminary results are also provided in this section. In Section 3, we establish the existence of at least one nontrivial solution of problem (1.1), by using a theorem of Bonanno and Bisci (see [10]). In Section 4, we establish a result of the existence of at least two nontrivial solutions of problem (1.1) by using a theorem of Bonanno and D'Aguí (see [7]). Finally, in Section 5, we prove the existence of at least three nontrivial solutions of problem (1.1), by using a theorem of Bonanno and Marano (see [9]).

2. Preliminaries

In this section, we first establish the variational framework associated with problem (1.1).
We consider the following T-dimensional Banach space.

$$S = \{u : \mathbb{Z}(0, T+1) \to \mathbb{R} \text{ such that } \triangle u(0) = \triangle u(T) = 0\}$$

equipped with the norm

$$\|u\| = \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} + \sum_{k=1}^{T} q(k)|u(k)|^{p^-}\right)^{1/p^-}.$$

On the space S we will also introduce the norm

$$\|u\|_{p^+} = \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} + \sum_{k=1}^{T} q(k)|u(k)|^{p^+}\right)^{1/p^+}$$

and the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\mu > 0 : \sum_{k=1}^{T+1} \left|\frac{\triangle u(k-1)}{\mu}\right|^{p(k-1)} + \sum_{k=1}^{T} q(k) \left|\frac{u(k)}{\mu}\right|^{p(k)} \leq 1\right\}.$$

We now recall the discrete weighted Hölder-type inequality (see [29]).

Lemma 2.1. *(Discrete weighted Hölder-type inequality) Let $\{a_k\}_{k=1}^N$, $\{b_k\}_{k=1}^N$ and $\{\omega_k\}_{k=1}^N$ be sequences of positive numbers.*
Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{k=1}^{N} \omega_k a_k b_k \leq \left(\sum_{k=1}^{N} \omega_k a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{N} \omega_k b_k^q\right)^{\frac{1}{q}}.$$

In the sequel, we will use the following inequality.

$$K\|u\|_{p^+} \leq \|u\| \leq 2^{\frac{p^+-p^-}{p^+p^-}} K\|u\|_{p^+}, \tag{2.1}$$

where $K := (\max\{T+1, Q\})^{\frac{p^+-p^-}{p^+p^-}}$. Indeed, by weighted Hölder's inequality (see Lemma 2.1), we get

$$\begin{aligned}
&\sum_{k=1}^{T} q(k)\, |u(k)|^{p^-} \\
&\leq \left(\sum_{k=1}^{T} q(k)(1)^{\frac{p^+}{p^+-p^-}}\right)^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T} q(k) \left(|u(k)|^{p^-}\right)^{\frac{p^+}{p^-}}\right)^{\frac{p^-}{p^+}} \\
&\leq Q^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T} q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}.
\end{aligned}$$

Using the same arguments, one has

$$\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} \leq (T+1)^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+}\right)^{\frac{p^-}{p^+}}.$$

Then, we get from the above inequalities and the fact that $\dfrac{p^-}{p^+} \leq 1$, the following.

$$\begin{aligned}
\|u\|^{p^-} &\leq (\max\{T+1, Q\})^{\frac{p^+-p^-}{p^+}} \\
&\times \left(\left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} \right)^{\frac{p^-}{p^+}} + \left(\sum_{k=1}^{T} q(k)|u(k)|^{p^+} \right)^{\frac{p^-}{p^+}} \right) \\
&\leq 2^{1-\frac{p^-}{p^+}} (\max\{T+1, Q\})^{\frac{p^+-p^-}{p^+}} \\
&\times \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} + \sum_{k=1}^{T} q(k)|u(k)|^{p^+} \right)^{\frac{p^-}{p^+}} = 2^{\frac{p^+-p^-}{p^+}} K^{p^-} \|u\|_{p^+}^{p^-}.
\end{aligned}$$

Therefore, $\|u\| \leq 2^{\frac{p^+-p^-}{p^+p^-}} K\|u\|_{p^+}$.
On the other hand, we get by the fact that $\dfrac{p^+}{p^-} \geq 1$, the following.

$$\begin{aligned}
\|u\|_{p^+}^{p^+} &\leq (\max\{T+1, Q\})^{\frac{p^--p^+}{p^-}} \\
&\times \left(\left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} \right)^{\frac{p^+}{p^-}} + \left(\sum_{k=1}^{T} q(k)|u(k)|^{p^-} \right)^{\frac{p^+}{p^-}} \right) \\
&\leq (\max\{T+1, Q\})^{\frac{p^--p^+}{p^-}} \\
&\times \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} + \sum_{k=1}^{T} q(k)|u(k)|^{p^-} \right)^{\frac{p^+}{p^-}} = K^{-p^+} \|u\|^{p^+}.
\end{aligned}$$

Therefore, $K\|u\|_{p^+} \leq \|u\|$. Thus, we obtain that (2.1) holds.
Moreover, we will also make use of the following norms.

$$\|u\|_\infty := \max\{|u(k)| : k \in \mathbb{Z}(1,T)\}, \text{ for all } u \in S.$$

For any $u \in S$, there exists $\tau \in \mathbb{Z}(1,T)$ such that

$$|u(\tau)|^{p^-} \leq \left| \sum_{k=\tau}^{T} u(k) \right|^{p^-} \leq \sum_{k=1}^{T} |u(k)|^{p^-},$$

then,

$$\begin{aligned}
\min_{k\in\mathbb{Z}(1,T)} q(k)|u(\tau)|^{p^-} &\leq \sum_{k=1}^{T} q(k)\,|u(k)|^{p^-} \\
&\leq \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} + \sum_{k=1}^{T} q(k)|u(k)|^{p^-}.
\end{aligned}$$

Therefore,

$$\|u\|_\infty \leq \frac{1}{\underline{q}^{1/p^-}}\|u\| \ \text{ for all } \ u \in S, \tag{2.2}$$

where $\underline{q} := \min_{k\in\mathbb{Z}(1,T)} q(k)$ (see [16]).

Since S is of finite dimension, the two last norms are equivalent. Therefore, there exist two constants $0 < L_1 < L_2$ such that

$$L_1\|u\|_{p(\cdot)} \leq \|u\| \leq L_2\|u\|_{p(\cdot)}. \tag{2.3}$$

Now, let $\varphi : S \to \mathbb{R}$ be defined by

$$\varphi(u) = \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)}. \tag{2.4}$$

As in [22], we have the following proposition.

Proposition 2.2. *If $u_n, u \in S$ then the following properties hold.*

$$\|u\|_{p(\cdot)} < 1 \ \Rightarrow \ \|u\|_{p(\cdot)}^{p^+} \leq \varphi(u) \leq \|u\|_{p(\cdot)}^{p^-}, \tag{2.5}$$

$$\|u\|_{p(\cdot)} > 1 \ \Rightarrow \ \|u\|_{p(\cdot)}^{p^-} \leq \varphi(u) \leq \|u\|_{p(\cdot)}^{p^+}, \tag{2.6}$$

$$\|u_n - u\|_{p(\cdot)} \to 0 \Leftrightarrow \varphi(u_n - u) \to 0 \ \text{ as } \ n \to \infty. \tag{2.7}$$

Definition 2.1. We say that $u \in S$ is a weak solution of problem (1.1) if

$$\sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1) + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k)$$

$$= \lambda \sum_{k=1}^{T} f(k, u(k))v(k),$$

for any $v \in S$.

Let us define the functionals $\Phi, \Psi : S \to \mathbb{R}$ as follows.

$$\Phi(u) = \Phi_1(u) + \Phi_2(u), \tag{2.8}$$

$$\Psi(u) := \sum_{k=1}^{T} \int_0^{u(k)} f(k,\tau)\, d\tau, \tag{2.9}$$

where

$$\Phi_1(u) := \sum_{k=1}^{T+1} \int_0^{|\triangle u(k-1)|} a(k-1,\xi)\xi\, d\xi, \quad \Phi_2(u) := \sum_{k=1}^{T} \frac{q(k)}{p(k)}|u(k)|^{p(k)}.$$

Assuming that for every $k \in \mathbb{Z}(1,T)$ and $t \in \mathbb{R}$,

$$F(k,\xi) = \int_0^{\xi} f(k,\tau)\, d\tau.$$

The energy functional associated to problem (1.1) is defined as $I_\lambda : S \to \mathbb{R}$,

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u). \tag{2.10}$$

Thus, it is easy to verify that Φ and Ψ are two functionals of class $C^1(S,\mathbb{R})$ whose Gâteaux derivatives at the point $u \in S$ are given by

$$\left\langle \Phi'(u), v \right\rangle = \sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1) + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k) \tag{2.11}$$

and

$$\left\langle \Psi'(u), v \right\rangle = \sum_{k=1}^{T} f(k, u(k))v(k), \tag{2.12}$$

for all $u, v \in S$.
We deduce by (2.11) and (2.12) that I_λ is of class $C^1(S,\mathbb{R})$ and its derivative is given by

$$\left\langle I'_\lambda(u), v \right\rangle = \left\langle \Phi'(u), v \right\rangle - \lambda \left\langle \Psi'(u), v \right\rangle,$$

for all $u, v \in S$. Since $\triangle u(0) = \triangle u(T) = 0$, we get

$$\sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1) = -\sum_{k=1}^{T} \triangle(a(k-1, |\triangle u(k-1)|)\triangle u(k-1))v(k),$$

then,

$$\left\langle I'_\lambda(u), v \right\rangle =$$

$$\sum_{k=1}^{T} \left[-\triangle(a(k-1, |\triangle u(k-1)|)\triangle u(k-1)) + q(k)|u(k)|^{p(k)-2}u(k) - \lambda f(k, u(k)) \right] v(k).$$

Thus, the critical points of I_λ are exactly the weak solutions of problem (1.1).

Now, we recall some auxiliary results to be used throughout the paper.

Lemma 2.3. (*a*) *Let $u \in S$ and $\|u\| > 1$. Then,*

$$\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \geq \|u\|^{p^-} - (1 + (1 + \overline{q})T).$$

(*b*) *Let $u \in S$ and $\|u\| < 1$. Then,*

$$\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \geq \frac{2^{\frac{p^- - p^+}{p^-}}}{K^{p^+}} \|u\|^{p^+}.$$

(*c*) *Let $u \in S$. Then,*

$$\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \leq \frac{1}{K^{p^+}} \|u\|^{p^+} + (1 + (1 + \overline{q})T).$$

Proof. Let $u \in S$ be fixed. By a similar argument as in [21], we define

$$\beta_k := \begin{cases} p^+ & \text{if } |\triangle u(k)| \leq 1 \\ p^- & \text{if } |\triangle u(k)| > 1 \end{cases} \quad \text{and} \quad \delta_k := \begin{cases} p^+ & \text{if } |u(k)| \leq 1 \\ p^- & \text{if } |u(k)| > 1, \end{cases}$$

for each $k \in \mathbb{Z}(1,T)$.
(a) For $u \in S$ with $\|u\| > 1$, one has

$$\begin{aligned}
&\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \\
&\geq \sum_{k=1,\beta_k=p^+}^{T+1} |\triangle u(k-1)|^{p^+} + \sum_{k=1,\beta_k=p^-}^{T+1} |\triangle u(k-1)|^{p^-} \\
&+ \sum_{k=1,\delta_k=p^+}^{T} q(k)\,|u(k)|^{p^+} + \sum_{k=1,\delta_k=p^-}^{T} q(k)\,|u(k)|^{p^-} \\
&= \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} - \sum_{k=1,\beta_k=p^+}^{T+1} \left(|\triangle u(k-1)|^{p^-} - |\triangle u(k-1)|^{p^+}\right) \\
&+ \sum_{k=1}^{T} q(k)|u(k)|^{p^-} - \overline{q} \sum_{k=1,\delta_k=p^+}^{T} \left(|u(k)|^{p^-} - |u(k)|^{p^+}\right) \\
&\geq \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} - (T+1) + \sum_{k=1}^{T} q(k)|u(k)|^{p^-} - \overline{q}T \\
&= \|u\|^{p^-} - (1 + (1+\overline{q})T).
\end{aligned}$$

(b) As $|\triangle u(k)| < 1$ and $|u(k)| < 1$ for each $k \in \mathbb{Z}(1,T)$ since $\|u\| < 1$, we deduce that

$$\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} \geq \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} \quad \text{and} \quad \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \geq \sum_{k=1}^{T} q(k)|u(k)|^{p^+}.$$

Hence, by the above inequalities and the relation (2.1), we obtain

$$\begin{aligned}
\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} &\geq \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} + \sum_{k=1}^{T} q(k)|u(k)|^{p^+} \\
&= \|u\|_{p^+}^{p^+} \\
&\geq \frac{2^{\frac{p^- - p^+}{p^-}}}{K^{p^+}} \|u\|^{p^+}.
\end{aligned}$$

(c) Indeed, we deduce by relation (2.1) that

$$\begin{aligned}
&\sum_{k=1}^{T+1}\Big[|\triangle u(k-1)|^{p(k-1)}+q(k)|u(k)|^{p(k)}\Big]\\
&\leq \sum_{k=1,\beta_k=p^-}^{T+1}|\triangle u(k-1)|^{p^+}+\sum_{k=1,\beta_k=p^+}^{T+1}|\triangle u(k-1)|^{p^-}\\
&+\sum_{k=1,\delta_k=p^-}^{T}q(k)\,|u(k)|^{p^+}+\sum_{k=1,\delta_k=p^+}^{T}q(k)\,|u(k)|^{p^-}\\
&=\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^+}+\sum_{k=1,\beta_k=p^+}^{T+1}\Big(|\triangle u(k-1)|^{p^-}-|\triangle u(k-1)|^{p^+}\Big)\\
&+\sum_{k=1}^{T}q(k)|u(k)|^{p^+}+\overline{q}\sum_{k=1,\delta_k=p^+}^{T}\Big(|u(k)|^{p^-}-|u(k)|^{p^+}\Big)\\
&\leq \sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^+}+(T+1)+\sum_{k=1}^{T+1}q(k)|u(k)|^{p^+}+\overline{q}T\\
&=\|u\|_{p^+}^{p^+}+(1+(1+\overline{q})T)\leq \frac{1}{K^{p^+}}\|u\|^{p^+}+(1+(1+\overline{q})T).
\end{aligned}$$

□

Proposition 2.4. *Under assumption $(H3)$, the following inequality holds true.*

$$\begin{aligned}
&\langle a(k,|u|)u-a(k,|v|)v,u-v\rangle\\
&\geq \begin{cases} c\,(|u|+|v|)^{p(k)-2}\,|u-v|^2 & \text{if } 1<p(k)<2\\ 4^{2-p^+}c|u-v|^{p(k)} & \text{if } \quad p(k)\geq 2\end{cases}
\end{aligned}\tag{2.13}$$

for all $u,v\in\mathbb{R}$ and $k\in\mathbb{Z}(1,T)$ such that $(u,v)\neq(0,0)$.

Proof. Let $u,v\in\mathbb{R}$ with $(u,v)\neq(0,0)$. Let us denote $\phi(k,u)=a(k,|u|)u$. Then, we have

$$\frac{\partial\phi(k,u)}{\partial u}=|u|\frac{\partial a}{\partial u}(k,|u|)+a(k,|u|),$$

for all $u\in\mathbb{R}\backslash\{0\}$. It follows by condition $(H3)$ that

$$\frac{\partial\phi(k,u)}{\partial u}\geq c|u|^{p(k)-2}.\tag{2.14}$$

Note that

$$\phi(k,u)-\phi(k,v)=\int_0^1\frac{\partial\phi(k,v+t(u-v))}{\partial u}(u-v)\,dt.\tag{2.15}$$

For $k\in\mathbb{Z}(0,T)$ such that $p(k)\geq 2$. By (2.14) and (2.15), we observe that

$$\begin{aligned}
\langle a(k,|u|)u-a(k,|v|)v,u-v\rangle &= \int_0^1\frac{\partial\phi}{\partial u}(k,v+t(u-v))(u-v)(u-v)\,dt\\
&\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2\,dt.
\end{aligned}$$

Without loss of generality, we can assume that $|u| \leq |v|$. Therefore, one has

$$|u-v| \quad \leq \quad |u|+|v| \Rightarrow |u-v| \leq 2|v|.$$

For $t \in [0, 1/4]$,

$$\begin{aligned} |v| = |v+t(u-v)-t(u-v)| &\leq |v+t(u-v)|+|t(u-v)| \\ &\leq |v+t(u-v)|+\frac{1}{4}|u-v|. \end{aligned}$$

Then,

$$|v+t(u-v)| \geq \frac{1}{4}|u-v|.$$

It follows that

$$\begin{aligned} \langle a(k,|u|)u-a(k,|v|)v, u-v\rangle &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2dt \\ &\geq 4^{2-p^+}c|u-v|^{p(k)}. \end{aligned}$$

For $k \in \mathbb{Z}(0,T)$ with $1 < p(k) < 2$. In the similar way as above, we obtain from condition $(H3)$ that

$$\frac{\partial\phi(k,u)}{\partial u} \geq c|u|^{p(k)-2},$$

for all $u \in \mathbb{R}\backslash\{0\}$.
In addition, let us consider $t \in [0,1]$, then $|tu+(1-t)v| \leq |u|+|v|$.
We get

$$|tu+(1-t)v|^{p(k)-2} \geq (|u|+|v|)^{p(k)-2}.$$

Therefore

$$\begin{aligned} \langle a(k,|u|)u-a(k,|v|)v, u-v\rangle &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2\,dt \\ &\geq c\,(|u|+|v|)^{p(k)-2}\,|u-v|^2. \end{aligned}$$

The proof is thus complete. ☐

Lemma 2.5. *Assume that $(H1)$ and $(H3)$ hold. Then, the operator $\Phi' : S \to S^*$ is strictly monotone on S and is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in S as $n \to \infty$ and $\limsup_{n\to\infty}\langle\Phi'(u_n)-\Phi'(u), u_n-u\rangle \leq 0$, then $u_n \to u$ in S as $n \to \infty$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between S and its dual S^*.*

Proof. We prove the strict monotonicity of Φ'. By using (2.13) and taking into account the well-known inequality, for any $x, y \in \mathbb{R}$,

$$\begin{aligned} &\left(|x|^{p(k)-2}x-|y|^{p(k)-2}y\right)(x-y) \\ &\geq \begin{cases} c_1\,(|x|+|y|)^{p(k)-2}\,|x-y|^2 & \text{if } 1<p(k)<2, \\ c_2|x-y|^{p(k)} & \text{if } \quad p(k)\geq 2, \end{cases} \end{aligned} \tag{2.16}$$

we can write for any $u, v \in S$ such that $u \neq v$,

$$\langle\Phi'(u)-\Phi'(v), u-v\rangle$$

$$\geq \begin{cases} \min\{c, c_1\} \displaystyle\sum_{k=1}^{T+1} \breve{u}(k-1)^{p(k-1)-2} |\triangle u(k-1) - \triangle v(k-1)|^2 > 0 \ \text{ if } \ 1 < p(k-1) < 2, \\ \min\left\{4^{2-p^+} c, c_2\right\} \displaystyle\sum_{k=1}^{T+1} |\triangle u(k-1) - \triangle v(k-1)|^{p(k-1)} > 0 \qquad \text{if} \qquad p(k-1) \geq 2, \end{cases}$$

where $\breve{u}(k-1) = |\triangle u(k-1)| + |\triangle v(k-1)|$. Consequently, Φ' is strictly monotone.

Now, we prove that the operator Φ' is of type (S_+). Let $\{u_n\}$ be a sequence in S such that $u_n \rightharpoonup u$ in S as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \leq 0.$$

We will show that $u_n \to u$ in S. By the above inequality and the strictly monotonicity of Φ', we get

$$\lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle = 0, \tag{2.17}$$

which means that

$$\lim_{n\to\infty} \left[\sum_{k=1}^{T+1} (a(k-1, |\triangle u_n(k-1)|)\triangle u_n(k-1) - a(k-1, |\triangle u(k-1)|)\triangle u(k-1)) \right.$$

$$\left. \times \triangle(u_n - u)(k-1) + \sum_{k=1}^{T} q(k) \left(|u_n(k)|^{p(k)-2} u_n(k) - |u(k)|^{p(k)-2} u(k) \right) (u_n - u)(k) \right] = 0.$$

Since $u_n \rightharpoonup u$ in S as $n \to \infty$, one has

$$\begin{cases} \displaystyle\lim_{n\to\infty} \sum_{k=1}^{T+1} (a(k-1, |\triangle u_n(k-1)|)\triangle u_n(k-1) - a(k-1, |\triangle u(k-1)|)\triangle u(k-1)) \\ \times(\triangle u_n(k-1) - \triangle u(k-1)) = 0, \\ \displaystyle\lim_{n\to\infty} \sum_{k=1}^{T} q(k) \left(|u_n(k)|^{p(k)-2} u_n(k) - |u(k)|^{p(k)-2} u(k) \right) (u_n(k) - u(k)) = 0. \end{cases} \tag{2.18}$$

Now, we prove that $\varphi(u_n - u) \to 0$ as $n \to \infty$. That is,

$$\lim_{n\to\infty} \sum_{k=1}^{T+1} |\triangle u_n(k-1) - \triangle u(k-1)|^{p(k-1)} = 0, \tag{2.19}$$

$$\lim_{n\to\infty} \sum_{k=1}^{T} q(k) |u_n(k) - u(k)|^{p(k)} = 0. \tag{2.20}$$

We first show (2.19). By (2.16) and Proposition 2.2, we have

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle$$
$$\geq \begin{cases} \min\{c, c_1\} \displaystyle\sum_{k=1}^{T+1} \hat{u}(k-1)^{p(k-1)-2} |\triangle u_n(k-1) - \triangle u(k-1)|^2 \ \text{ if } \ 1 < p(k-1) < 2, \\ \min\left\{4^{2-p^+} c, c_2\right\} \displaystyle\sum_{k=1}^{T+1} |\triangle u_n(k-1) - \triangle u(k-1)|^{p(k-1)} \qquad \text{if} \qquad p(k-1) \geq 2, \end{cases} \tag{2.21}$$

with $\hat{u}(k-1) = |\triangle u_n(k-1)| + |\triangle u(k-1)|$.

By using the discrete Hölder inequality (see [22]), we know that

$$\begin{aligned}
&\sum_{k=1}^{T+1} |\triangle u_n(k-1) - \triangle u(k-1)|^{p(k-1)} \\
&\quad = \sum_{k=1}^{T+1} \hat{u}(k-1)^{\frac{p(k-1)(2-p(k-1))}{2}} \left(\hat{u}(k-1)^{\frac{p(k-1)(p(k-1)-2)}{2}} |\triangle u_n(k-1) - \triangle u(k-1)|^{p(k-1)}\right) \\
&\quad \le 2\|\hat{u}^{\frac{p(\cdot)(2-p(\cdot))}{2}}\|_{\frac{2}{2-p(\cdot)}} \|\hat{u}^{\frac{p(\cdot)(p(\cdot)-2)}{2}} |\triangle u_n(k-1) - \triangle u(k-1)|^{p(\cdot)}\|_{\frac{2}{p(\cdot)}} \\
&\quad \le 2\|\hat{u}\|_{p(\cdot)}^{\sigma} \left(\sum_{k=1}^{T+1} \hat{u}(k-1)^{p(k-1)-2} |\triangle u_n(k-1) - \triangle u(k-1)|^2\right)^{\omega}, \qquad (2.22)
\end{aligned}$$

where σ is either $p^-(2-\tilde{p})/2$ or $\tilde{p}(2-p^-)/2$ and ω is either $p^-/2$ or $\tilde{p}/2$ with $\tilde{p} = \sup_{\{k\in\mathbb{Z}:1<p(k)<2\}} p(k)$. Then, by (2.17), (2.21) and (2.22), it follows that

$$\lim_{n\to\infty} \sum_{k=1}^{T+1} |\triangle u_n(k-1) - \triangle u(k-1)|^{p(k-1)} = 0. \qquad (2.23)$$

Next, we will show (2.20). We suppose that $k \in \mathbb{Z}(0,T)$ such that $p(k) \ge 2$. For any $u \in S$, $\{u_n\} \subset S$, we get by (2.16) that

$$\left(|u_n(k)|^{p(k)-2} u_n(k) - |u(k)|^{p(k)-2} u(k)\right)(u_n(k) - u(k)) \ge c_2 |u_n(k) - u(k)|^{p(k)}.$$

Then, summing up k from 1 to T, one obtains

$$\begin{aligned}
&\sum_{k=1}^{T} q(k) \left(|u_n(k)|^{p(k)-2} u_n(k) - |u(k)|^{p(k)-2} u(k)\right)(u_n(k) - u(k)) \\
&\ge c_2 \sum_{k=1}^{T} q(k) |u_n(k) - u(k)|^{p(k)}. \qquad (2.24)
\end{aligned}$$

It follows from (2.18) and (2.24) that

$$\lim_{n\to\infty} \sum_{k=1}^{T} q(k) |u_n(k) - u(k)|^{p(k)} = 0. \qquad (2.25)$$

Next, for $k \in \mathbb{Z}(0,T)$ such that $1 < p(k) < 2$, from (2.16), we see that

$$\begin{aligned}
&\sum_{k=1}^{T} q(k) \left(|u_n(k)|^{p(k)-2} u_n(k) - |u(k)|^{p(k)-2} u(k)\right)(u_n(k) - u(k)) \\
&\ge c_1 \sum_{k=1}^{T} q(k) \check{u}(k)^{p(k)-2} |u_n(k) - u(k)|^2, \qquad (2.26)
\end{aligned}$$

with $\breve{u}(k) = |u_n(k)| + |u(k)|$.

By using the discrete Hölder inequality (see [22]), we get

$$\begin{aligned}
&\sum_{k=1}^{T} q(k)|u_n(k) - u(k)|^{p(k)} \\
= &\sum_{k=1}^{T} q(k)\breve{u}(k)^{\frac{p(k)(2-p(k))}{2}} \left(\breve{u}(k)^{\frac{p(k)(p(k)-2)}{2}} |u_n(k) - u(k)|^{p(k)}\right) \\
\leq \; & 2\|q(k)\breve{u}^{\frac{p(\cdot)(2-p(\cdot))}{2}}\|_{\frac{2}{2-p(\cdot)}} \|q(k)\breve{u}^{\frac{p(\cdot)(p(\cdot)-2)}{2}} |u_n(k) - u(k)|^{p(\cdot)}\|_{\frac{2}{p(\cdot)}} \\
\leq \; & 2\|\breve{u}\|_{p(\cdot)}^{\nu} \left(\sum_{k=1}^{T} q(k)\breve{u}(k)^{p(k)-2}|u_n(k) - u(k)|^2\right)^{\varsigma}, \qquad (2.27)
\end{aligned}$$

where ν is either $p^-(2-\hat{p})/2$ or $\hat{p}(2-p^-)/2$ and ς is either $p^-/2$ or $\hat{p}/2$ with $\hat{p} = \max\limits_{\{k\in\mathbb{Z}(0,T):1<p(k)<2\}} p(k)$. Thus, it follows from (2.18), (2.26) and (2.27) that

$$\lim_{n\to\infty} \sum_{k=1}^{T} q(k)|u_n(k) - u(k)|^{p(k)} = 0. \qquad (2.28)$$

Relations (2.23), (2.25), (2.28) combined with (2.7) imply that $\|u_n - u\|_{p(\cdot)} \to 0$ and thus $\|u_n - u\| \to 0$ as $n \to \infty$. This proves that Φ' is of type (S_+). The proof of Lemma 2.3 is complete. □

Lemma 2.6. *Under assumptions* $(H1)$ *and* $(H3)$, *the functional* $\Phi : S \to \mathbb{R}$ *is weakly lower semicontinuous, i.e.,* $u_n \rightharpoonup u$ *in* S *as* $n \to \infty$ *implies that* $\Phi(u) \leq \liminf\limits_{n\to\infty} \Phi(u_n)$.

Proof. Assume that $u_n \rightharpoonup u$ in S as $n \to \infty$. It follows from (2.11) and Lemma 2.3 that Φ is convex (see [39, Proposition 42.6]) and therefore for any $n \in \mathbb{N}$,

$$\Phi(u_n) \geq \Phi(u) + \langle \Phi'(u), u_n - u\rangle.$$

Then,

$$\liminf_{n\to\infty} \Phi(u_n) \geq \Phi(u) + \liminf_{n\to\infty}\langle \Phi'(u), u_n - u\rangle = \Phi(u).$$

Hence, Φ is weakly lower semicontinuous. The proof is complete. □

3. Existence of at least one nontrivial solution of (1.1)

In this section, one uses the following result due to Bonanno and Bisci (see [10]).

Theorem 3.1. [10] *Let* X *be a reflexive real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two Gâteaux differentiable functionals such that* Φ *is (strongly) continuous, sequentially weakly lower semicontinuous and coercive in* X *and* Ψ *is sequentially weakly upper semicontinuous in* X. *Let* I_λ *be the functional defined by* $I_\lambda := \Phi - \lambda\Psi$,

$\lambda \in \mathbb{R}$ and for each $r > \inf_{u\in X} \Phi(u)$, let ψ be the function defined by

$$\psi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}.$$

Then, for each $r > \inf_{u\in X} \Phi(u)$ and each $\lambda \in (0, \frac{1}{\psi(r)})$, the restriction of I_λ to $\Phi^{-1}((-\infty, r))$ has a global minimum.

The main result of this section is the following.

Theorem 3.2. *Let ε be a positive constant and assume that $f(k, 0) \neq 0$ for some $k \in \mathbb{Z}(1, T)$. Then, for any*

$$\lambda \in \left(0, \frac{\left(\underline{q}^{1/p^-}\right)^{p^*} \min\{1, c\}}{p^+ L^{p^*}} \frac{\varepsilon^{p^*}}{\sum_{k=1}^{T} \max_{|\xi|\le\varepsilon} F(k,\xi)}\right),$$

problem (1.1) has at least one nontrivial solution $u \in S$ such that $\|u\|_\infty < \varepsilon$.

Proof. Let us apply Theorem 3.1 by choosing $X := S$, and put Φ, Ψ and I_λ as in (2.8), (2.9) and (2.10), respectively. We know from (2.11), (2.12) and Lemma 2.4 that Φ is a continuously differentiable and sequentially weakly lower semicontinuous in S and Ψ is a continuously differentiable functional and is sequentially weakly upper semicontinuous in S. Note that the critical points of I_λ are exactly the solutions of problem (1.1).

Next, let $\|u\| > 1$. Using the condition $(H3)$ and Lemma 2.2(a), one has

$$\begin{aligned}
\Phi(u) &= \sum_{k=1}^{T+1} \int_0^{|\triangle u(k-1)|} a(k-1,\xi)\xi\, d\xi + \sum_{k=1}^{T} \frac{q(k)}{p(k)} |u(k)|^{p(k)} \\
&\geq \frac{c}{p^+} \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \frac{1}{p^+} \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \\
&\geq \frac{1}{p^+} \min\{1, c\} \left(\|u\|^{p^-} - (1 + (1+\overline{q})T)\right) \to \infty \text{ as } \|u\| \to \infty.
\end{aligned}$$

Hence, Φ is coercive and Φ and Ψ satisfy all regularity assumptions requested in Theorem 3.1.

Put

$$r = \frac{\left(\underline{q}^{1/p^-}\right)^{p^*} \min\{1, c\}\, \varepsilon^{p^*}}{p^+ L^{p^*}} \tag{3.1}$$

and define

$$\alpha^{p*} := \begin{cases} \alpha^{p^-} & \text{if } \alpha > 1, \\ \alpha^{p^+} & \text{if } \alpha < 1 \end{cases} \qquad \text{and} \quad L := \begin{cases} L_1 & \text{if } \|u\| < L_1, \\ L_2 & \text{if } \|u\| > L_2. \end{cases} \tag{3.2}$$

Clearly, we have $r > \inf_{u\in S} \Phi(u)$. Moreover, for all $u \in S$ such that $\Phi(u) < r$, from (2.3), (2.4), (2.5), (2.6) and (3.2), one has

$$\begin{aligned} r > \Phi(u) &= \sum_{k=1}^{T+1} \int_0^{|\triangle u(k-1)|} a(k-1,\xi)\xi \, d\xi + \sum_{k=1}^{T} \frac{q(k)}{p(k)} |u(k)|^{p(k)} \\ &\geq \frac{c}{p^+} \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \frac{1}{p^+} \sum_{k=1}^{T} q(k) |u(k)|^{p(k)} \\ &\geq \frac{\min\{1,c\}\, \varphi(u)}{p^+} \geq \frac{\min\{1,c\}\, \|u\|_{p(\cdot)}^{p^*}}{p^+} \geq \frac{\min\{1,c\}\, \|u\|^{p^*}}{p^+ L^{p^*}}, \end{aligned}$$

and so

$$\|u\| \leq L \left(\frac{rp^+}{\min\{1,c\}} \right)^{1/p^*}. \tag{3.3}$$

By (2.2) and (3.3), we can write

$$\|u\|_\infty \leq \underline{q}^{-1/p^-} \|u\| \leq \underline{q}^{-1/p^-} L \left(\frac{rp^+}{\min\{1,c\}} \right)^{1/p^*} := \varepsilon.$$

Therefore,

$$\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v) = \sup_{\Phi(v)<r} \Psi(v) = \sup_{\Phi(v)<r} \sum_{k=1}^{T} F(k, v(k)) \leq \sum_{k=1}^{T} \max_{|\xi|\leq\varepsilon} F(k,\xi).$$

By the definition of ψ, since $0_X \in \Phi^{-1}((-\infty, r))$ and $\Phi(0_X) = \Psi(0_X) = 0$, one has

$$\begin{aligned} \psi(r) &= \inf_{\Phi(u)<r} \frac{\left(\sup_{\Phi(v)<r} \Psi(v) \right) - \Psi(u)}{r - \Phi(u)} \leq \frac{\sup_{\Phi(v)<r} \Psi(v)}{r} \\ &= \frac{\sup_{\Phi(v)<r} \sum_{k=1}^{T} F(k, v(k))}{r} \leq \frac{p^+ L^{p^*}}{\left(\underline{q}^{1/p^-}\right)^{p^*} \min\{1,c\}} \frac{\sum_{k=1}^{T} \max_{|\xi|\leq\varepsilon} F(k,\xi)}{\varepsilon^{p^*}}. \end{aligned}$$

Hence, it follows that

$$\lambda < \frac{\left(\underline{q}^{1/p^-}\right)^{p^*} \min\{1,c\}}{p^+ L^{p^*}} \frac{\varepsilon^{p^*}}{\sum_{k=1}^{T} \max_{|\xi|\leq\varepsilon} F(k,\xi)} \leq \frac{1}{\psi(r)},$$

that is,

$$\lambda \in \left(0, \frac{\left(\underline{q}^{1/p^-}\right)^{p^*} \min\{1,c\}}{p^+ L^{p^*}} \frac{\varepsilon^{p^*}}{\sum_{k=1}^{T} \max_{|\xi|\leq\varepsilon} F(k,\xi)} \right) \subset \left(0, \frac{1}{\psi(r)} \right).$$

Therefore, the functional I_λ admits a non-zero critical point $u \in S$ such that $\Phi(u) < r$. $\square$

4. Existence of at least two nontrivial solutions of (1.1)

In this section, one uses the following theorem due to Bonanno and D'Aguí (see [7]).

Theorem 4.1. [7] *Let X be a real finite dimensional Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

(i) $\sigma = \dfrac{1}{r} \sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) < \dfrac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \rho,$

(ii) for each $\lambda \in \Lambda := (\frac{1}{\rho}, \frac{1}{\sigma})$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

Let $f : \mathbb{Z}(1,T) \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following hypothesis.

$$(H4) \quad \liminf_{|\xi|\to\infty} \frac{\displaystyle\int_0^\xi f(k,\tau)\,d\tau}{|\xi|^{p^+}} \geq 0, \quad \text{for any } k \in \mathbb{Z}(1,T).$$

We will prove that I_λ satisfies the Palais-Smale condition and unbounded from below.

Lemma 4.2. *Assume that $(H4)$ holds. Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, \infty)$, the functional I_λ satisfies the Palais-Smale condition and is unbounded from below.*

Proof. Fix $\lambda - \lambda^* > 0$. Let $\{u_n\} \subset S$ be a sequence with $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Since S is a finite-dimensional space, it is sufficient to verify that $\{u_n\}$ is bounded. Assume by contradiction that $\{u_n\}$ is unbounded. Then, passing to a subsequence, one has $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may assume that $\|u_n\| > 1$ for any $n \in \mathbb{N}$.

Since $\liminf_{|\xi|\to\infty} \dfrac{\int_0^\xi f(k,\tau)\,d\tau}{|\xi|^{p^+}} \geq 0$, for $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_0^\xi f(k,\tau)\,d\tau \geq \epsilon|\xi|^{p^+} \text{ for all } k \in \mathbb{Z}(1,T) \text{ and all } \xi \in \mathbb{R} \text{ with } |\xi| > \delta.$$

Since $\xi \to \int_0^\xi f(k,\tau)\,d\tau - \epsilon|\xi|^{p^+}$ is continuous on $[-\delta, \delta]$, there exists $C_\delta > 0$ such that

$$\int_0^\xi f(k,\tau)\,d\tau - \epsilon|\xi|^{p^+} \geq -C_\delta \text{ for all } k \in \mathbb{Z}(1,T) \text{ and all } \xi \in [-\delta, \delta]\,.$$

Consequently, we infer that

$$\int_0^{\xi} f(k,\tau)\,d\tau \geq \epsilon|\xi|^{p^+} - C_\delta \text{ for all } (k,\xi) \in \mathbb{Z}(1,T)\times\mathbb{R}. \tag{4.1}$$

So, by using relation (4.1), assumption $(H1)$ and relation (c) of Lemma 2.2, we deduce that

$$\begin{aligned}
I_\lambda(u_n) &= \sum_{k=1}^{T+1}\int_0^{|\triangle u_n(k-1)|} a(k-1,\xi)\xi\,d\xi + \sum_{k=1}^{T}\frac{q(k)}{p(k)}|u_n(k)|^{p(k)} - \lambda\sum_{k=1}^{T}\int_0^{u_n(k)} f(k,\tau)\,d\tau \\
&\leq \overline{a_1}\sum_{k=1}^{T+1}|\triangle u_n(k-1)| + \frac{a_2}{p^-}\sum_{k=1}^{T+1}|\triangle u_n(k-1)|^{p(k-1)} + \frac{1}{p^-}\sum_{k=1}^{T}q(k)|u_n(k)|^{p(k)} \\
&- \lambda\epsilon\sum_{k=1}^{T}|u_n(k)|^{p^+} + \lambda C_\delta T \\
&\leq 2\overline{a_1}\sum_{k=1}^{T}|u_n(k)| + \frac{\max\{1,a_2\}}{p^-}\left(\frac{\|u_n\|^{p^+}}{K^{p^+}} + (1+(1+\overline{q})T)\right) \\
&- \lambda\epsilon\sum_{k=1}^{T}|u_n(k)|^{p^+} + \lambda C_\delta T.
\end{aligned} \tag{4.2}$$

Note that

$$\begin{aligned}
\|u_n\|^{p^-} &\leq 2^{p^--1}\sum_{k=1}^{T}\Big(|u_n(k+1)|^{p^-} + |u_n(k)|^{p^-}\Big) + \overline{q}\sum_{k=1}^{T}|u_n(k)|^{p^-} \\
&\leq 2^{p^-}\sum_{k=1}^{T}|u_n(k)|^{p^-} + \overline{q}\sum_{k=1}^{T}|u_n(k)|^{p^-} \\
&\leq \Big(2^{p^-}+\overline{q}\Big)T^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T}|u_n(k)|^{p^+}\right)^{\frac{p^-}{p^+}}.
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{T}|u_n(k)|^{p^+} \geq \frac{\|u_n\|^{p^+}}{T^{\frac{p^+-p^-}{p^+}}\big(2^{p^-}+\overline{q}\big)^{\frac{p^+}{p^-}}}. \tag{4.3}$$

Then, we get by (2.2), (4.2) and (4.3) that

$$\begin{aligned}
I_\lambda(u_n) &\leq 2\overline{a_1}T\|u\|_\infty + \frac{\max\{1,a_2\}}{p^-}\left(\frac{\|u_n\|^{p^+}}{K^{p^+}} + (1+(1+\overline{q})T)\right) \\
&- \lambda\epsilon\frac{\|u_n\|^{p^+}}{T^{\frac{p^+-p^-}{p^+}}\left(2^{p^-}+\overline{q}\right)^{\frac{p^+}{p^-}}} + \lambda C_\delta T \\
&\leq 2\overline{a_1}\underline{q}^{-1/p^-}T\|u_n\| + \left(\frac{\max\{1,a_2\}}{p^-K^{p^+}} - \lambda\epsilon\frac{1}{T^{\frac{p^+-p^-}{p^+}}\left(2^{p^-}+\overline{q}\right)^{\frac{p^+}{p^-}}}\right)\|u_n\|^{p^+} \\
&+ \frac{\max\{1,a_2\}}{p^-}(1+(1+\overline{q})T) + \lambda C_\delta T.
\end{aligned}$$

Thus, if we choose

$$\lambda^* := \frac{\max\{1,a_2\}T^{\frac{p^+-p^-}{p^+}}\left(2^{p^-}+\overline{q}\right)^{\frac{p^+}{p^-}}}{\varepsilon p^- K^{p^+}},$$

then for any $\lambda \in (\lambda^*, \infty)$, $I_\lambda(u_n) \to -\infty$ since $\|u_n\| \to \infty$ and this is a contradiction. Hence, $\{u_n\}$ is bounded and so I_λ satisfies the Palais-Smale condition for all $\lambda \in (\lambda^*, \infty)$.

It remains to show that I_λ is unbounded from below. Suppose that $\{u_n\}$ is unbounded. By assumption $(H3)$ and relation (a) of Lemma 2.2, one has

$$I_\lambda(u_n) \geq \frac{1}{p^+}\min\{1,c\}\left(\|u_n\|^{p^-} - (1+(1+\overline{q})T)\right).$$

Since $\|u_n\| \to \infty$, then $I_\lambda(u_n) \to \infty$. This is a contradiction by the fact that $\{I_\lambda(u_n)\}$ is bounded. It follows that the sequence $\{u_n\}$ is bounded. Hence, the proof is complete. □

The existence result immediately follows.
We can now give the existence result of at least two nontrivial solutions of problem (1.1).

Theorem 4.3. *Assume that there exist two positive constants b and ε with*

$$\varepsilon > \frac{\left\{\max\left\{b, b^{p^*}\right\}\right\}^{\frac{1}{p^*}}(p^+)^{\frac{1}{p^*}}L\left(2\overline{a_1} + \frac{1}{p^-}\max\{1,a_2\}(2+Q)\right)^{\frac{1}{p^*}}}{\underline{q}^{1/p^-}\{\min\{1,c\}\}^{\frac{1}{p^*}}} \tag{4.4}$$

such that

$$\frac{\sum_{k=1}^{T}\max_{|\xi|\leq\varepsilon}F(k,\xi)}{\frac{\left(\underline{q}^{1/p^-}\right)^{p^*}\min\{1,c\}\,\varepsilon^{p^*}}{p^+L^{p^*}}} < \min\left\{\frac{\sum_{k=1}^{T}F(k,b)}{\max\{b,b^{p^*}\}\left(2\overline{a_1} + \frac{1}{p^-}\max\{1,a_2\}(2+Q)\right)}, \frac{1}{\lambda^*}\right\}. \tag{4.5}$$

Then, for any

$$\lambda \in \left(\max\left\{ \frac{\max\left\{b, b^{p^*}\right\}\left(2\overline{a_1} + \frac{1}{p^-}\max\{1, a_2\}(2+Q)\right)}{\sum_{k=1}^{T} F(k,b)}, \lambda^* \right\}, \frac{\frac{\left(\underline{q}^{1/p^-}\right)^{p^*}\min\{1,c\}\,\varepsilon^{p^*}}{p^+ L^{p^*}}}{\sum_{k=1}^{T}\max_{|\xi|\le\varepsilon} F(k,\xi)} \right),$$

problem (1.1) *admits at least two nontrivial solutions.*

Proof. We know that Φ and Ψ are well-defined and continuously Gâteaux differentiable. Clearly, by the definitions of Φ and Ψ, one has

$$\inf_{u\in X}\Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Note that by Lemma 4.1, the functional I_λ satisfies the Palais-Smale condition for any $\lambda > \lambda^*$ and it is unbounded from below. On the other hand, put

$$\tilde{u}(k) := \begin{cases} b & \text{if } k \in \mathbb{Z}(1,T), \\ 0 & \text{otherwise.} \end{cases}$$

So, we deduce that $\tilde{u} \in S$, $\Psi(\tilde{u}) = \sum_{k=1}^{T} F(k, \tilde{u}(k)) = \sum_{k=1}^{T} F(k,b)$ and

$$\begin{aligned} \Phi(\tilde{u}) &\le \overline{a_1}\sum_{k=1}^{T+1}|\triangle\tilde{u}(k-1)| + \frac{a_2}{p^-}\sum_{k=1}^{T+1}|\triangle\tilde{u}(k-1)|^{p(k-1)} + \frac{1}{p^-}\sum_{k=1}^{T} q(k)|\tilde{u}(k)|^{p(k)} \\ &\le \overline{a_1}(b+b) + \frac{1}{p^-}\max\{1,a_2\}\left(b^{p(0)} + b^{p(T)} + \sum_{k=1}^{T} q(k) b^{p(k)}\right) \\ &\le 2\overline{a_1}b + \frac{b^{p^*}}{p^-}\max\{1,a_2\}\left(2 + \sum_{k=1}^{T} q(k)\right) \\ &\le \max\left\{b, b^{p^*}\right\}\left(2\overline{a_1} + \frac{1}{p^-}\max\{1,a_2\}(2+Q)\right). \end{aligned}$$

Since

$$\varepsilon > \frac{\left\{\max\left\{b, b^{p^*}\right\}\right\}^{\frac{1}{p^*}}(p^+)^{\frac{1}{p^*}} L\left(2\overline{a_1} + \frac{1}{p^-}\max\{1,a_2\}(2+Q)\right)^{\frac{1}{p^*}}}{\underline{q}^{1/p^-}\left\{\min\{1,c\}\right\}^{\frac{1}{p^*}}},$$

we obtain $\Phi(\tilde{u}) < r$, where r is as in (3.1). Moreover, one has

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \ge \frac{\sum_{k=1}^{T} F(k,b)}{\max\{b, b^{p^*}\}\left(2\overline{a_1} + \frac{1}{p^-}\max\{1,a_2\}(2+Q)\right)}. \tag{4.6}$$

For all $u \in S$ such that $\Phi(u) < r$, taking (2.2) and (3.3) into account, one has

$$\|u\|_\infty \le \underline{q}^{-1/p^-} \|u\| \le \underline{q}^{-1/p^-} L \left(\frac{rp^+}{\min\{1, c\}} \right)^{1/p^*} := \varepsilon.$$

Therefore,

$$\Phi^{-1}((-\infty, r]) \subseteq \{u \in S \text{ such that } \|u\|_\infty \le \varepsilon\}.$$

Thus, one has

$$\frac{\sup\limits_{u \in \Phi^{-1}((-\infty,r])} \Psi(u)}{r} \le \frac{\sum\limits_{k=1}^{T} \max\limits_{|\xi| \le \varepsilon} F(k, \xi)}{\dfrac{\left(\underline{q}^{1/p^-}\right)^{p^*} \min\{1, c\} \varepsilon^{p^*}}{p^+ L^{p^*}}}. \tag{4.7}$$

Taking (4.5), (4.6) and (4.7) into account, one has

$$\sigma = \frac{1}{r} \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \rho.$$

Therefore, the assertion (i) of Theorem 4.1 follows.

Now, from Lemma 4.1, for each

$$\lambda \in \Lambda := (\frac{1}{\rho}, \frac{1}{\sigma}),$$

the functional I_λ satisfies the Palais-Smale condition and it is unbounded from below. Consequently, the assertion (ii) of Theorem 4.1 follows. Then, all the hypotheses of Theorem 4.1 hold, and I_λ has at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2} \in S$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$ for all $\lambda \in \Lambda$, which are nontrivial solutions of problem (1.1). □

5. Existence of at least three nontrivial solutions of (1.1)

In this section, one uses the following theorem due to Bonanno and Marano (see [9]).

Theorem 5.1. [9] *Let X be a reflexive real Banach space and let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\inf\limits_{u \in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\tilde{u} \in X$, with $r < \Phi(\tilde{u})$, such that*

(i) $\sigma = \dfrac{1}{r} \sup\limits_{\Phi(u) \le r} \Psi(u) < \dfrac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \rho,$

(ii) for each $\lambda \in \Lambda_r := (\frac{1}{\rho}, \frac{1}{\sigma})$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional I_λ has at least three nontrivial critical points.

One makes the following additional assumption on the function $f : \mathbb{Z}(1, T) \times \mathbb{R} \to \mathbb{R}$ as follows.

$$(H5)\ \limsup_{|\xi|\to\infty} \frac{\displaystyle\int_0^\xi f(k,\tau)\,d\tau}{|\xi|^{p^-}} \le 0, \text{ for any } k \in \mathbb{Z}(1,T).$$

Lemma 5.2. *Let $p^- > 1$. Assume that the condition $(H5)$ holds. Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the functional I_λ is coercive.*

Proof. Let $\|u\| > 1$. By assumption $(H3)$, we deduce that

$$\begin{aligned} I_\lambda(u) &= \sum_{k=1}^{T+1} \int_0^{|\triangle u(k-1)|} a(k-1,\xi)\xi\,d\xi + \sum_{k=1}^{T} \frac{q(k)}{p(k)}|u(k)|^{p(k)} - \lambda \sum_{k=1}^{T} F(k,u(k)) \\ &\ge \frac{\min\{1,c\}}{p^+}\left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)}\right) \\ &- \lambda \sum_{k=1}^{T} \int_0^{u(k)} f(k,\tau)\,d\tau. \end{aligned} \tag{5.1}$$

By $(H5)$, for $\epsilon_1 > 0$, there exists $\rho > 0$ such that

$$\int_0^\xi f(k,\tau)\,d\tau \le \epsilon_1 |\xi|^{p^-} \text{ for all } k \in \mathbb{Z}(1,T) \text{ and } \xi \in \mathbb{R} \text{ with } |\xi| > \rho.$$

Since $\xi \to \displaystyle\int_0^\xi f(k,\tau)\,d\tau - \epsilon_1|\xi|^{p^-}$ is continuous on $[-\rho,\rho]$, there exists $C_\rho > 0$ such that

$$\int_0^\xi f(k,\tau)\,d\tau - \epsilon_1|\xi|^{p^-} \le C_\rho \text{ for all } k \in \mathbb{Z}(1,T) \text{ and all } \xi \in [-\rho,\rho].$$

Thus,

$$\int_0^\xi f(k,\tau)\,d\tau \le \epsilon_1|\xi|^{p^-} + C_\rho \text{ for all } (k,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R}. \tag{5.2}$$

Now, note that by (2.2), one has

$$|u(k)|^{p^-} \le \frac{1}{\underline{q}}\|u\|^{p^-} \text{ for all } k \in \mathbb{Z}(1,T).$$

Then, summing up for k goes from 1 to T, it follows

$$\sum_{k=1}^{T} |u(k)|^{p^-} \le \frac{T}{\underline{q}}\|u\|^{p^-}. \tag{5.3}$$

By (5.2) and (5.3), we get

$$\sum_{k=1}^{T} \int_0^{u(k)} f(k,\tau)\,d\tau \le \frac{\epsilon_1 T}{\underline{q}}\|u\|^{p^-} + C_\rho T.$$

For $u \in S$ such that $\|u\| > 1$, the above estimation combined with (5.1) and Lemma 2.2(a), gives

$$\begin{aligned} I_\lambda(u) &\geq \frac{\min\{1,c\}}{p^+}\left(\|u\|^{p^-} - (1+(1+\overline{q})T)\right) - \lambda\frac{\epsilon_1 T}{\underline{q}}\|u\|^{p^-} - \lambda C_\rho T \\ &\geq \left(\frac{\min\{1,c\}}{p^+} - \lambda\frac{\epsilon_1 T}{\underline{q}}\right)\|u\|^{p^-} - \left(\frac{\min\{1,c\}}{p^+}(1+(1+\overline{q})T) + \lambda C_\rho T\right). \end{aligned}$$

Choosing

$$\lambda^* = \frac{\underline{q}\min\{1,c\}}{p^+\epsilon_1 T},$$

we obtain that $I_\lambda(u) \to \infty$ as $\|u\| \to \infty$. We conclude that for any $\lambda \in (0,\lambda^*)$, I_λ is coercive. $\square$

We have the following result.

Theorem 5.3. *Let $p^- > 1$. Assume that there exist two positive constants c_d and d with $d > \dfrac{\underline{q}^{1/p^-}\varepsilon}{L}$, such that*

$$\frac{\displaystyle\sum_{k=1}^{T}\max_{|\xi|\leq\varepsilon}F(k,\xi)}{\dfrac{\left(\underline{q}^{1/p^-}\right)^{p^*}\min\{1,c\}\,\varepsilon^{p^*}}{p^+L^{p^*}}} < \min\left\{\frac{c_d\displaystyle\sum_{k=1}^{T}F(k,d)}{\dfrac{d^{p^*}}{p^+}\min\{1,c\}\,(2+Q)}, \frac{1}{\lambda^*}\right\}.$$

Then, for any

$$\lambda \in \left(\max\left\{\frac{\dfrac{d^{p^*}}{p^+}\min\{1,c\}\,(2+Q)}{c_d\displaystyle\sum_{k=1}^{T}F(k,d)}, \lambda^*\right\}, \frac{\dfrac{\left(\underline{q}^{1/p^-}\right)^{p^*}\min\{1,c\}\,\varepsilon^{p^*}}{p^+L^{p^*}}}{\displaystyle\sum_{k=1}^{T}\max_{|\xi|\leq\varepsilon}F(k,\xi)}\right),$$

problem (1.1) *admits at least three nontrivial solutions.*

Proof. We know that Φ and Ψ are well defined and continuously Gâteaux differentiable, and $\inf\limits_{u\in X}\Phi(u) = \Phi(0) = \Psi(0) = 0$. Furthermore, by Lemma 2.4, Φ is sequentially weakly lower semicontinuous, while Proposition 1 of [11] ensures that Φ' admits a continuous inverse on X^*. Form Lemma 5.1, the assertion (ii) of Theorem 5.1 follows.

In order to prove (i) of Theorem 5.1, we consider $\tilde{v} \in S$ defined as follows.

$$\tilde{v}(k) := \begin{cases} d & \text{if } k \in \mathbb{Z}(1,T), \\ 0 & \text{if } k = 0 \text{ or } k = T+1. \end{cases}$$

So, we deduce that

$$\begin{aligned}\Phi(\tilde{v}) &\geq \frac{c}{p^+}\left(d^{p(0)}+d^{p(T)}\right)+\frac{1}{p^+}\sum_{k=1}^{T} q(k)d^{p(k)}\\ &\geq \frac{1}{p^+}\min\{1,c\}\left(d^{p(0)}+d^{p(T)}+\sum_{k=1}^{T} q(k)d^{p(k)}\right)\\ &\geq \frac{d^{p^*}}{p^+}\min\{1,c\}(2+Q).\end{aligned}$$

Hence, from $d>\dfrac{\underline{q}^{1/p^-}\varepsilon}{L}$, we can write $\Phi(\tilde{v})>r$, where r is as in (3.1). Moreover, one has

$$\frac{\Psi(\tilde{v})}{\Phi(\tilde{v})}\leq \frac{\sum_{k=1}^{T} F(k,d)}{\dfrac{d^{p^*}}{p^+}\min\{1,c\}(2+Q)}.$$

For all $u\in\Phi^{-1}((-\infty,r])$, similarly to (4.7), one has

$$\frac{\sup\limits_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{r}\leq \frac{\sum_{k=1}^{T}\max\limits_{|\xi|\leq\varepsilon} F(k,\xi)}{\dfrac{\left(\underline{q}^{1/p^-}\right)^{p^*}\min\{1,c\}\,\varepsilon^{p^*}}{p^+L^{p^*}}}.$$

Therefore, (i) of Theorem 5.1 follows. Thus, all the assumptions of Theorem 5.1 are fulfilled, and then, for all $\lambda\in\Lambda$, the functional I_λ has at least three nontrivial critical points, which are three nontrivial solutions of problem (1.1). $\square$

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Received 13 December 2024

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